

# Nonfibered knots and representation shifts

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## Abstract

A conjecture of [13] states that a knot is nonfibered if and only if its infinite cyclic cover has uncountably many finite covers. We prove the conjecture for a class of knots that includes all knots of genus 1, using techniques from symbolic dynamics.

*Keywords:* Knot, knot group, representation shift. <sup>1</sup>

## 1 Introduction

Let  $G$  be a finitely presented group with epimorphism  $\chi : G \rightarrow \mathbb{Z}$ . The kernel  $K$  of  $\chi$  need not be finitely generated. However,  $K$  is finitely presented as a  $\mathbb{Z}$ -operator group [11]. In [12] [13] the authors exploited this structure to show that the representations of  $K$  into a fixed finite group  $\Sigma$  form a *shift of finite type*, a simple dynamical system described by a finite directed graph. We call this dynamical system the *representation shift* of  $K$  in  $\Sigma$ . When  $G$  is a knot or link group, representation shifts inform us about the algebraic topology of finite covering spaces from a purely dynamical perspective.

We review basic definitions of representation shifts and give a partial solution to Conjecture 4.4 of [13]. The complete solution would characterize nonfibered knots as knots with complicated representation shifts, where complexity is measured by topological entropy.

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## 2 Review of Representation Shifts

An *augmented group system* [7] is a triple  $\mathcal{G} = (G, \chi, x)$  consisting of a finitely presented group  $G$ , epimorphism  $\chi : G \rightarrow \mathbb{Z}$  and distinguished element  $x \in G$  such that  $\chi(x) = 1$ . Two such systems  $\mathcal{G}_i = (G_i, \chi_i, x_i)$ ,  $i = 1, 2$  are *equivalent* (and regarded as the same) if there exists an isomorphism  $f : G_1 \rightarrow G_2$  such that  $f(x_1) = x_2$  and  $\chi_1 = \chi_2 \circ f$ .

**Example 2.1.** An augmented group system is associated to an oriented knot  $k \subset \mathbb{S}^3$  in a canonical manner. Let  $G = \pi_1(\mathbb{S}^3 \setminus k, p)$ , where the base point  $p$  is contained on the boundary  $\partial N(k)$  of a tubular neighborhood  $N(k) = \mathbb{S}^1 \times \mathbb{D}^2$  of  $k$ . Let  $x$  be the homotopy class of a meridian  $m \subset \partial N(k)$ , with orientation acquired from  $k$ . Finally, let  $\chi : G \rightarrow \mathbb{Z}$  be the abelianization homomorphism that sends  $x$  to 1. It follows from the uniqueness of tubular neighborhoods that  $\mathcal{G} = (G, \chi, x)$  is well defined.

We denote the kernel of  $\chi$  by  $K$ . Given any finite group  $\Sigma$ , we consider the space  $\text{Hom}(K, \Sigma)$  of representations  $\rho : K \rightarrow \Sigma$ . The basis for its topology is given by the sets

$$\mathcal{N}_{a_1, \dots, a_s}(\rho) = \{\rho' \mid \rho'(a_i) = \rho(a_i), i = 1, \dots, s\},$$

where  $a_1, \dots, a_s$  varies over all finite collections of elements of  $K$ . The topology is the compact-open topology where  $K$  and  $\Sigma$  are discrete spaces. Roughly speaking, representations are close in  $\text{Hom}(K, \Sigma)$  if they agree on large finitely generated subgroups of  $K$ . The distinguished element  $x$  induces a self-map  $\sigma_x$  of  $\text{Hom}(K, \Sigma)$  defined by

$$\sigma_x \rho(a) = \rho(x^{-1}ax) \quad \forall a \in K.$$

It is easily seen that  $\sigma_x$  is a homeomorphism.

The *representation shift* associated to  $\mathcal{G} = (G, \chi, x)$  and  $\Sigma$  is the pair  $(\text{Hom}(K, \Sigma), \sigma_x)$ . We denote it by  $\Phi_\Sigma(\mathcal{G})$ . It is a dynamical system well defined up to topological conjugacy [11]. More precisely, if  $\mathcal{G}_i$ ,  $i = 1, 2$ , are equivalent augmented group systems, then there exists a homeomorphism  $F$  of the underlying spaces of  $\Phi_\Sigma(\mathcal{G}_i)$  such that  $F \circ \sigma_{x_1} = \sigma_{x_2} \circ F$ .

The representation shift  $\Phi_\Sigma(\mathcal{G})$  is an example of a *shift of finite type*, a special type of expansive 0-dimensional dynamical system, one that can be described by a finite directed graph. (See [4].) We use combinatorial group theory to construct such a graph for a representation shift.

Given an augmented group system  $\mathcal{G} = (G, \chi, x)$ , we can describe  $G$  as an HNN extension  $\langle x, B \mid x^{-1}ax = \phi(a), \forall a \in U \rangle$ , where  $B$  is a finitely

generated subgroup of  $K$ , and  $U, V$  are isomorphic finitely generated subgroups of  $B$  with isomorphism  $\phi : U \rightarrow V$  (see [6]). The subgroup  $B$  is an *HNN base*. One can choose  $B$  so that it contains any prescribed finite subset of  $K$  (see [10]).

**Example 2.2.** Let  $\mathcal{G} = (G, \chi, x)$  be an augmented group system associated to a knot, as in Example 2.1. An HNN decomposition for  $G$  can be obtained in a natural way. Begin with a  $\pi_1$ -incompressible Seifert surface for  $k$  meeting the exterior  $E(k) = \mathbb{S}^3 \setminus \text{int } N(k)$  in a connected surface  $S$ . Let  $(W; S_0, S_1)$  be the resulting cobordism, with boundary comprising two copies  $S_0, S_1$  of  $S$  joined by an annulus  $\partial S \times I$ . Let  $B = \pi_1(W, p)$ , where the basepoint  $p$  lies on the boundary of  $S_0$ . Let  $U = \pi_1(S_0, p)$ . The meridian  $m$  appears as a path from  $p \in S_0$  to a point  $p_1 \in S_1$ . Use the path to regard  $\pi_1(S_1, p_1)$  as a subgroup  $V$  of  $B$ . Clearly  $G$  is described as  $(B; U, V, \phi)$ , where  $\phi$  is induced by the gluing of  $S_0$  to  $S_1$  when recovering the exterior  $E(k)$ .

Conjugation by  $x$  induces an automorphism of  $K$ . Let  $B_j = x^{-j} B x^j$ ,  $U_j = x^{-j} U x^j$  and  $V_j = x^{-j} V x^j$ , for  $j \in \mathbb{Z}$ . Then  $K$  is described as an infinite amalgamated free product

$$K = \langle B_j \mid V_j = U_{j+1}, \forall j \in \mathbb{Z} \rangle.$$

The vertex set of the graph  $\Gamma$  consists of all representations  $\rho_0 : U \rightarrow \Sigma$ , a finite set since  $U$  is finitely generated. If  $\bar{\rho}_0$  is a representation from  $B$  to  $\Sigma$ , then we draw a directed edge labeled  $\bar{\rho}_0$  from the vertex  $\rho_0 = \bar{\rho}_0|_U$  to the vertex  $\rho'_0 = \bar{\rho}_0|_V \circ \phi$ . ( $\Gamma$  may have parallel edges.) Consider a bi-infinite path in  $\Gamma$  given by an edge sequence

$$\cdots \bar{\rho}_{-2} \bar{\rho}_{-1} \bar{\rho}_0 \bar{\rho}_1 \bar{\rho}_2 \cdots$$

The representations  $B_j \rightarrow \Sigma$  given by  $a \mapsto \bar{\rho}_j(x^j a x^{-j})$  have a unique common extension  $\rho : K \rightarrow \Sigma$ . Conversely, any representation  $\rho : K \rightarrow \Sigma$  arises from such a path, and uniquely. Thus bi-infinite paths of the graph  $\Gamma$  correspond bijectively to elements of  $\text{Hom}(K, \Sigma)$ . The map  $\sigma_x$  acts as the left coordinate shift on the sequence of edges.

We may “prune”  $\Gamma$  by removing any vertex or edge that is not contained in a bi-infinite path. The resulting graph has finitely many bi-infinite paths iff it consists of a collection of disjoint cycles. It contains uncountably many bi-infinite paths iff it contains two cycles with at least one common vertex.

A representation  $\rho \in \Phi_\Sigma(\mathcal{G})$  has *period*  $r$  if  $\sigma_x^r(\rho) = \rho$ . Such representations correspond to closed paths in  $\Gamma$  with length dividing  $r$ . The set of

representations with period  $r$  is denoted by  $\text{Fix}(\sigma_x^r)$ . If  $M_r$  is the  $r$ -fold cyclic cover of  $\mathbb{S}^3$  branched over a knot  $k$ , then  $\text{Fix}(\sigma_x^r)$  is in natural bijective correspondence with  $\text{Hom}(\pi_1 M_r, \Sigma)$  [13]. This correspondence connects dynamical properties of the representation shift with topological properties of  $k$ .

Topological entropy is one measure of complexity for a dynamical system. For a shift of finite type, it can be computed as the log of the spectral radius of the adjacency matrix  $A$  of any directed graph that describes the shift. (Here  $A_{i,j}$  is the number of edges from the  $i$ th vertex to the  $j$ th.) Consequently, the topological entropy of  $\Phi_\Sigma(\mathcal{G})$ , denoted by  $h_\Sigma(\mathcal{G})$ , is the exponential growth rate of  $|\text{Hom}(\pi_1 M_r, \Sigma)|$  (see [13]). Notice that if  $K$  is finitely generated, then  $\Phi_\Sigma(\mathcal{G})$  is finite for all  $\Sigma$ , and so in this case  $h_\Sigma(\mathcal{G})=0$ .

Let  $S_N$  denote the symmetric group on  $\{1, \dots, N\}$ . It is well known that elements  $\rho \in \text{Hom}(K, S_N)$  correspond in a finite-to-one manner with subgroups  $H \leq K$  with index no greater than  $N$ . The correspondence is

$$\rho \mapsto \{g \in K \mid \rho(g)(1) = 1\}.$$

The preimage of a subgroup of index  $N$  consists of  $(N-1)!$  transitive representations. (A representation  $\rho$  is transitive if  $\rho(K)$  operates transitively on  $\{1, \dots, N\}$ .) Note that if  $\Phi_{S_N}(\mathcal{G})$  is uncountable, then  $K$  contains uncountably many subgroups of some index no greater than  $N$ . Hence the infinite cyclic cover of  $k$  has uncountably many finite covers.

We summarize the results of this section. Recall that any finite group embeds in a sufficiently large symmetric group.

**Proposition 2.3.** *Let  $k \subset \mathbb{S}^3$  be a knot with associated augmented group system  $\mathcal{G}$ . Then the following statements are equivalent.*

- (1) *The infinite cyclic cover of  $k$  has uncountably many finite covers.*
- (2) *The representation shift  $\Phi_\Sigma(\mathcal{G})$  is uncountable, for some finite group  $\Sigma$ .*
- (3) *The topological entropy  $h_\Sigma(\mathcal{G})$  is positive, for some finite group  $\Sigma$ .*
- (4)  *$\lim_{r \rightarrow \infty} \frac{1}{r} \log |\text{Hom}(\pi_1 M_r, \Sigma)|$  is positive, for some finite group  $\Sigma$ .*

### 3 Nonfibered knots

We recall that a knot  $k \subset \mathbb{S}^3$  is fibered if its exterior  $E(k) = \mathbb{S}^3 \setminus \text{int } N(k)$  fibers over the circle. It is no loss of generality to assume that the fibration restricts to the standard projection  $\partial N(k) \simeq k \times \mathbb{S}^1 \rightarrow \mathbb{S}^1$ . Hence  $E(k)$  is

seen to be homeomorphic to a mapping torus  $S \times I/F$ , where  $F : S \rightarrow S$  is a homeomorphism of a minimal-genus Seifert surface  $S$  of  $k$ .

If  $k$  is fibered, then the commutator subgroup  $G'$  of its group is finitely generated and free, isomorphic to  $\pi_1 S$ . Conversely, a theorem of J. Stallings [15] implies that if  $k$  is a knot such that  $G'$  is finitely generated, then in fact  $G'$  is free and  $k$  is fibered.

If  $k$  is fibered and  $\mathcal{G}$  is its associated augmented group system, then for any finite group  $\Sigma$ , the representation shift  $\Phi_\Sigma(\mathcal{G})$  is finite. Its order is  $|\Sigma|^{2g}$ , where  $g$  is the genus of  $k$  (equal to the genus of its fiber). The trefoil and figure-eight knots are the only fibered knots of genus 1.

Conjecture 4.4 of [13] proposes a characterization of nonfibered knots. It states that  $k$  is nonfibered iff the entropy  $h_\Sigma(\mathcal{G})$  is positive for some finite group  $\Sigma$ .

**Remark 3.1.** (1) In terms of the HNN base  $B$  described above, the condition that  $k$  is not fibered is equivalent to the condition that  $U$  is a proper subgroup of  $B$ . Lemma 2.3 (Substitution Lemma) of [12] provides a strategy for showing that some  $\Phi_{S_n}(\mathcal{G})$  is uncountable: Find a periodic element of  $\Phi_{S_N}(\mathcal{G})$  such that some symbol, say  $N$ , is fixed by every permutation in the image of  $U$  but moved by some element of  $\rho(K)$ . Recall that periodic representations correspond to cycles in the graph  $\Gamma$ . By introducing a new symbol (enlarging  $S_N$  to  $S_{N+1}$ ), we can construct another periodic representation corresponding to a second cycle, one that branches from the first. Then  $\Phi_{S_{N+1}}(\mathcal{G})$  is uncountable.

(2) For our strategy, it suffices to find any representation  $\tilde{\rho} : G \rightarrow \Sigma$  such that  $\rho(U)$  is a proper subgroup of  $\rho(K)$ . For given such a representation, and letting  $\rho : K \rightarrow \Sigma$  be the restriction, we enumerate the cosets of  $\rho(U)$  in  $\rho(K)$ , say  $1, \dots, N$  ( $N > 1$ ). In a natural way,  $\rho$  determines an element of  $\Phi_{S_N}(\mathcal{G})$ :  $a \in K$  is sent to the transitive permutation of cosets given by right multiplication by  $\rho(a)$ . Note that if  $a \in U$ , then such a permutation fixes the symbol corresponding to  $\rho(U)$ . Finally, we note that if  $r$  is the order of  $\tilde{\rho}(x)$  in  $\Sigma$ , then  $\sigma_x^r \rho = \rho$ , since  $(\sigma_x^r \rho)(a) = \rho(x^{-r} a x^r) = \tilde{\rho}(x^{-1})^r \rho(a) \tilde{\rho}(x)^r = \rho(a)$ , for all  $a \in K$ .

The representation  $\tilde{\rho}$  in the Remark 3.1 (2) “separates” the subgroup  $U$  from some element  $a \in K$ .

In general, a subgroup  $U$  of a group  $G$  is *separable* if for any element  $a \in G \setminus U$ , there exists a finite-index subgroup of  $G$  that contains  $U$  but not  $a$ . Equivalently, there exists a finite representation  $\tilde{\rho} : G \rightarrow \Sigma$  such that  $\tilde{\rho}(a) \notin \tilde{\rho}(U)$ . The strategy outlined in Remark 3.1(2) requires only that  $U$  can be separated from *some* element of  $K \setminus U$ .

**Definition 3.2.** An element  $a \in G \setminus U$  is *separable from  $U$*  if there exists a subgroup  $H$  of finite index in  $G$  containing  $U$  but not  $a$ .

Question 15 of [16] asks if any finitely generated subgroup of a finitely-generated Kleinian group is separable. An affirmative answer would establish Theorem 3.4 for all hyperbolic knots. Although Thurston's question remains open, a result of D. Long and G. Niblo [5] enables us to apply our strategy in the case of genus-1 knots (see also remarks that follow).

The theorem of Long and Niblo has been used by S. Friedl and S. Vidussi in [1] to show that twisted Alexander polynomials corresponding to finite representations decide if a genus-1 knot is fibered.

**Theorem 3.3.** (D. Long and G. Niblo [5]) *Let  $M$  be an orientable Haken 3-manifold. If  $i : T \hookrightarrow M$  is an incompressibly embedded torus, then  $i_*(\pi_1 T)$  is separable in  $\pi_1 M$ .*

**Theorem 3.4.** *Let  $k$  be a knot of genus 1. Then  $k$  is nonfibered iff the conclusions of Proposition 2.3 hold.*

*Proof.* One implication of the theorem is clear: if the conclusion of Proposition 2.3 holds, then  $k$  is nonfibered.

Assume that  $k$  is nonfibered. Consider the 3-manifold  $M$  obtained by 0-framed surgery on  $k$ ; that is, by removing and replacing a tubular neighborhood  $N(k) \equiv k \times \mathbb{D}^2$  in such a way that each disk  $* \times \mathbb{D}^2$  bounds a longitude of  $k$ . By results of [3],  $M$  is irreducible. We denote the fundamental group of  $M$  by  $\hat{G}$ .

The addition of a meridional disk converts a genus-1 Seifert surface  $S$  for  $k$  to a torus  $\hat{S}$  in  $M$ . Since  $\hat{S}$  is dual to a nontrivial cohomology class and  $M$  is irreducible, we see that  $\hat{S}$  is incompressible. Note in particular that  $M$  is Haken.

Obtain an HNN decomposition  $(\hat{B}; \hat{U}, \hat{V})$  for  $\hat{G}$  much as we did for  $G$ , by splitting  $M$  along  $\hat{S}$ . Here  $\hat{U} = \pi_1 \hat{S}$ . Since  $k$  is not fibered, neither is  $M$  [2]. Hence  $\hat{U}$  must be a proper subgroup of  $\hat{B}$ . Select an element  $\hat{a} \in \hat{B} \setminus \hat{U}$ . By Theorem 3.3 there exists a finite group  $\Sigma$  and homomorphism  $\hat{\rho} : \hat{G} \rightarrow \Sigma$  such that  $\hat{\rho}(\hat{a}) \notin \hat{U}$ .

The group  $\hat{G}$  is a quotient of  $G$ . Let  $p$  be the natural projection. Note that  $p(U) = \hat{U}$ . Choose  $a \in K$  such that  $p(a) = \hat{a}$ . Define  $\rho = \hat{\rho} \circ p : G \rightarrow \Sigma$ .

Remark 3.1(2) completes the proof.  $\square$

Genus-1 knots are plentiful, the simplest examples being the twist knots (e.g. the knots  $5_2, 6_1$ ) and doubled knots (obtained from a knot and any push-off by joining with a clasp). We extend the collection of nonfibered

knots with uncountable representation shifts by considering also any knot  $k$  with group  $G$  that maps homomorphically onto the group  $\bar{G}$  of a nonfibered genus-1 knot  $\bar{k}$ . Examples of such knots  $k$  include satellite knots with genus-1 pattern knot [8].

**Corollary 3.5.** *Let  $k$  be a knot. Assume that the group of  $k$  maps onto the group of a nonfibered knot  $\bar{k}$  of genus 1. Then  $k$  is nonfibered and the conclusions of Proposition 2.3 hold.*

*Proof.* Assume that  $h : G \rightarrow \bar{G}$  is an epimorphism, where  $G, \bar{G}$  are the groups of  $k, \bar{k}$ , respectively. Let  $K, \bar{K}$  denote the respective commutator subgroups, and  $x, \bar{x}$  the meridional generators of  $k, \bar{k}$ .

Since  $h(K) = \bar{K}$  and  $\bar{K}$  is not finitely generated, we see at once that  $K$  is not finitely generated. Hence  $k$  is nonfibered.

If  $h(x) = \bar{x}$ , then for any finite group  $\Sigma$ , the representation shift  $\Phi_\Sigma(\bar{\mathcal{G}})$  corresponding to  $\bar{k}$  is a subshift of the representation shift  $\Phi_\Sigma(\mathcal{G})$  corresponding to  $k$ ; that is,  $\text{Hom}(\bar{K}, \Sigma)$  is a subspace of  $\text{Hom}(K, \Sigma)$  with the shift map  $\sigma_x$  restricting to  $\sigma_{\bar{x}}$ . The epimorphism  $h$  induces an embedding:  $h^*\rho = \rho \circ h$ . It follows that the topological entropy  $h_\Sigma(\mathcal{G})$  is no less than  $h_\Sigma(\bar{\mathcal{G}})$ . By theorem 3.4,  $h_\Sigma(\bar{\mathcal{G}}) > 0$  for some finite group  $\Sigma$ . Hence for such a group,  $h_\Sigma(\mathcal{G})$  is also positive.

If  $h(x) \neq \bar{x}$ , then there exists  $a \in K$  such that  $h(ax) = \bar{x}^\epsilon$ , where  $\epsilon = \pm 1$ . We may assume without loss of generality that  $\epsilon = 1$ . In this case, we replace  $x$  by  $ax$ . Of course the augmented group system  $\mathcal{G}$  and associated representation shifts  $\Phi_\Sigma(\mathcal{G})$  change. However, by a result of [11], the topological entropy of the representation shift remains unchanged. As in the case in which  $h(x) = \bar{x}$ , there exists a finite group  $\Sigma$  such that  $h_\Sigma(\mathcal{G}) > 0$ .  $\square$

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